March 1, 2023

Week 8

2020B Adv. Cal II

We give ideas behind the proof of the formula

$$(*) \quad \iint_{D_2} f(x,y)\, dA(x,y) = \iint_{D_1} \hat{f}(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$$

when $\hat{f}(u,v) = f(x(u,v), y(u,v))$, and $\Phi(u,v) = (x(u,v), y(u,v))$ maps the interior of D_1 1-1 onto the interior of D_2.

First of all, we need two facts:

(1) The linear part / first approximation of a function $f(x)$ at x_0 is the linear function $f(x_0) + f'(x_0)(x - x_0)$.

Ideas: Taylor expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

As $\frac{f''(x_0)}{2}(x - x_0)^2$ is v. small compared to $f(x_0) + f'(x_0)(x - x_0)$ when $\Delta x = x - x_0$ is v. small, we can approximate $f(x)$ by $f(x_0) + f'(x_0)(x - x_0)$.

For $f(x,y)$ at (x_0, y_0), Taylor expansion becomes

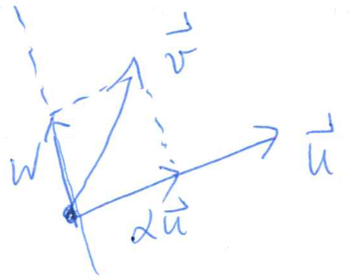
$$f(x,y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right) + \frac{1}{2!} \left(\frac{\partial^2 f}{\partial x^2} \Delta x^2 + 2 \frac{\partial f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial f}{\partial y^2} \Delta y^2 \right) + \cdots$$

where the partial derivatives evaluate at (x_0, y_0).

We take $f(x_0, y_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$ to be the first order approximation of $f(x, y)$ at (x_0, y_0).

Similar ideas work for vector-valued functions.

(2) Let $\vec{u}$ and $\vec{v}$ be 2 vectors in the plane, we claim the area of the parallelogram spanned by these vectors is the absolute value of the determinant $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$.



Let

$$\vec{v} = \alpha\vec{u} + \vec{w}, \quad \vec{w} \perp \vec{u}$$

Then

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ \alpha u_1 + w_1 & \alpha u_2 + w_2 \end{vmatrix}$$

$$= u_1 w_2 - u_2 w_1$$

$$= (u_1, u_2) \cdot (w_2, -w_1)$$

Since $\vec{z} = (w_2, -w_1)$ satisfies $\vec{z} \cdot \vec{w} = 0$,

$\vec{z}$ is perpendicular to $\vec{w}$, hence the angle θ between $\vec{z}$ and $\vec{u}$ is either $0°$ or π. So

$$\vec{u} \cdot \vec{z} = |\vec{u}||\vec{z}| \cos\theta$$

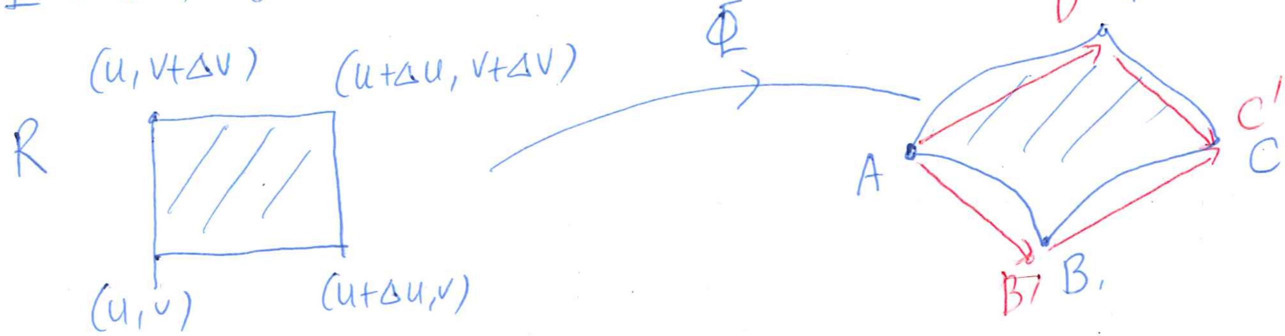
$$= \pm |\vec{u}||\vec{z}|$$

$$= \pm |\vec{u}||\vec{w}|.$$

Observing that the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ is $|\vec{u}||\vec{w}|$, got it.

Return to the pf of (*)

Let R be a small rectangle sitting inside D_1. Its image under Φ is $\mathcal{D}$, say.



A $(x(u,v), y(u,v))$

B $(x(u+\Delta u, v), y(u+\Delta u, v))$

C $(x(u, v+\Delta v), y(u, v+\Delta v))$

D $(x(u+\Delta u, v+\Delta v), y(u+\Delta u, v+\Delta v))$

$$x(u+\Delta u, v) = x(u,v) + \frac{\partial x}{\partial u}\Delta u + \cdots$$
$$\cong x(u,v) + x_u \Delta u$$
$$y(u+\Delta u, v) = y(u,v) + \frac{\partial y}{\partial u}\Delta u + \cdots$$
$$\cong y(u,v) + y_u \Delta u.$$

So, by replacing B by B' (formed by its first approx.)

$$B'(x(u,v) + x_u \Delta u,\ y(u,v) + y_u \Delta u).$$

Similarly,

$$C \to C'(x(u,v) + x_u\Delta u + x_v \Delta v,\ y(u,v) + y_u \Delta u + y_v \Delta v)$$
$$D \to D'(x(u,v) + x_v \Delta v,\ y(u,v) + y_v \Delta v)$$

The area of $\mathcal{D}$ is approximately the area of the paralogram $AB'C'D'$. Using Fact (2), its area is

$$\begin{vmatrix} x_u \Delta u & y_u \Delta u \\ x_v \Delta v & y_v \Delta v \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v.$$

Since the area of R is $\Delta u \Delta v$, we see that the area ratio between R and $\mathcal{D}$ is approximately,

$$\frac{\frac{\partial(x,y)}{\partial(u,v)}\Delta u \Delta v}{\Delta u \Delta v} = \frac{\partial(x,y)}{\partial(u,v)}$$

(should take absolute value)

Now, let P be a partition on D_1 (assume it is a rectangle). Under Φ, the subrectangle R_{ij} go to $\mathcal{D}_{ij}$. We see that

$$\iint_{D_2} f(x,y)\,dA(x,y) = \sum_{i,j} \iint_{\mathcal{D}_{ij}} f(x,y)\,dA(x,y)$$

$$\approx \sum_{i,j} f(P_{ij})\,|\mathcal{D}_{ij}|, \qquad P_{ij} \in \mathcal{D}_{ij} \text{ tag pt}$$

$$= \sum_{i,j} f(\Phi(q_{ij}))\,|\mathcal{D}_{ij}| \qquad \Phi(q_{ij}) = P_{ij}.$$

$$= \sum_{i,j} \hat{f}(q_{ij})\,|\mathcal{D}_{ij}|$$

$$= \sum_{i,j} \hat{f}(q_{ij}) \frac{|\mathcal{D}_{ij}|}{\Delta u_i \Delta v_j} \Delta u_i \Delta v_j$$

As $\|P\| \to 0$, $\Delta u_i, \Delta v_j$ v. small

$$\approx \sum_{i,j} \hat{f}(q_{ij}) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| (u_i, v_j)\, \Delta u_i v_j$$

Now, take $q_{ij} = (u_i, v_j)$ to be the tag,

$$= \sum_{i,j} \hat{f}(u_i, v_j) \left| \frac{\partial(x,y)}{\partial(u,v)} (u_i, v_j) \right| \Delta u_i v_j$$

$$\to \iint_{D_1} \hat{f}(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v). \qquad \#$$

(this lecture is for optional reading)